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A Stochastic Maximum Principle and Cox, Ingersoll, Ross Interest Rate Model for an Optimal Investment under Partial Information

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we solve the problem of hedging of a European option. This is done by determining the optimal strategy for investing in the risky asset constituting the portfolio. We assume that the payoff Π of this contingent asset at maturity date T does not only depend on the price S_T of this risky asset to which it relates. But it also depends on an unobservable random variable B. In addition to the risky asset, the portfolio for this hedging contains an asset whose price S_t^0 on each date t before maturity is a deterministic function of a stochastic short-term interest rate r_t. The dynamics of the process $(r_t)_{0\leq t\leq T}$ is that of the CIR (Cox, Ingersoll and Ross) model. which does not give negative values according to the parameters of the said model. We use classical filtering theory and stochastic partial differential equations (SPDE). Thus, we move from the partial information to the full information. Then, we use a stochastic maximum principle that we established with the backward stochastic differential equations (BSDE) to determine the optimal investment strategies $(\hat{u}_t)_{0\leq t\leq T}$ and $(\hat{u}_t^0)_{0\leq t\leq T}$ in the risky asset respectively in the presence of the option and in its absence.

1. Introduction

Nowadays, we witness an appalling development of risk management and profitability products in the financial markets. Along with the development of pricing methods for calculating the prices of derivative products, we are witnessing the development of hedging techniques for the coverage of these contingent assets. In fact, since the economic upheavals of the $1970s$ with the abandonment of the Bretton Woods system, the deregulation of financial markets by the desire for globalization, the global economic and financial environment is now facing macroeconomic imbalances and large variations in interest rates, exchange rates with the oil shocks and increased business risks. We are now experiencing strong volatility in interest rates, exchange rates, and commodity prices. This global economic instability is marked by various crises, some of which are: The 1973 oil crisis, the subprime crisis of 2008 and the economic crisis of Covid even if it is more social. Thus, alongside traditional financial products we are witnessing a development of specialized derivative assets. With regard to these specialized contingent assets which are risk hedging tools, alongside the vanilla options such as European calls and puts, we now find exotic options. Next to the main places financial institutions such as NYSE (New York Stock Exchange) there are specialized financial markets.

Derivative products have taken an increasingly important place at the global level in the profile risk and profitability of financial institutions. They are used by financial institutions both as risk management tools and as sources of income.

If the use of options is an effective instrument for risk management, calculate its price and covering its requirements is a difficult fact for its issuer given the risks it faces transferred to the payoff of uncertain character. Several authors have tried to solve this problem (see [6, 14, 17, 22, 24, 27, 29, 30, 34]). In incomplete but viable markets, in the absence of strategy replicating an option, one is led for the management of the portfolio either to the maximization the utility of the terminal wealth of the portfolio (see [6, 12, 22, 24, 30]), or to the minimization of related risk (see [4, 23]).

In this paper, we solve the problem of hedging of a European option with maturity T on a risky financial asset which at any date $t \in [0,T]$ the price is S_t. The stochastic interest rate r_t on the market is

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described with CIR model. We assume that the option's payoff Π depends on the price of the observable risky asset S_T and an unobservable random variable Z_T which represents all uncertain events related to the market environment. As in [6, 12, 24, 30] using the improved Black-Scholes model, we assume that a factor μ of the drift of the risky asset is a function of both the interest rate r_t and the unobservable variable Z_t. However here, we consider the general case where μ is not only an affine function of these two variables but some deterministic function. This is also contrary to certain articles where it is rather volatility which is the unobservable variable.

The value $X_t^{x,u}$ of our portfolio at date t consists of, on the one hand, the price S_t of the risky asset and the S_t^0 price of an asset which is a deterministic function of the stochastic interest rate.

We propose ourselves determine the optimal investment strategy $(\hat{u}_t)_{0\leq t\leq T}$ representing the optimal quantity of risky assets to invest in the portfolio in the presence of the option and the same strategy $(\hat{u}_t^0)_{0\leq t\leq T}$ in the absence of the option. The optimal control $(\hat{u}_t)_{0\leq t\leq T}$ represents at each instant t the quantity of risky asset to invest in a portfolio under the condition of maximizing the utility of the terminal wealth of the portfolio and must be adapted to the natural filtration generated by the observable variables: S_t, r_t. This being seen as a problem under full information. Unfortunately, the direct resolution of this problem can lead to the fact that this optimal control also depends on the unobservable variable $(Z_t)_{0\leq t\leq T}$ which makes it a problem under partial information. We assume that our economy agent is in a buy and sell situation. With a financial portfolio model of Black Scholes, we base our hedging strategy on maximizing of the utility of wealth terminal of the portfolio. The use of the unobservable variable Z_t leads to considering the problem as being under partial information.

The authors in [6], [12], [30] have considered this problem but with a constant interest rate or deterministic and as the only observable variable the price of the risky asset S_t. Here we consider a stochastic interest rate and two observable variables. The following section 2 presents more the problem, the motivations, the model, the method of resolution and some definitions. Section 3 is devoted to the passage of the problem under partial information to the problem under complete information thanks to Girsanov's theorem. Indeed, any control strategy $(\hat{u}_t)_{0\leq t\leq T}$ must be adapted to the σ –algebra generated by the observable variables (S_t, and r_t). In section 4, we present a stochastic maximum principle established, necessary for determining optimal strategies. This is done using the Hamiltonian and the adjoint equations. Section 5 is devoted to the application of the previous tools to the resolution of the problem.

2. Motivations, problem and model

In a financial market faced with several assets, we must choose the optimal quantity to invest in each while taking into account the risks linked to the market environment and the expected gain. We consider here the problem of management of a portfolio comprising two financial assets whose prices at date $t \in [0, T]$ are respectively S_t and S_t^0. The risky asset is of price S_t and the one we qualify as semi-risky of price S_t^0 with a stochastic short-term interest rate r_t, unlike the authors in [30] and [6] where it is deterministic or constant.

In this work, we consider the problem of an economic agent who sells at price x an option European payoff Π on a risky asset and wishes to cover this payoff with an optimal strategy investment in this risky asset. For a European option of horizon T on the risky asset subject to basic risks and uncertain B with the payoff $\Pi(S_T, B)$ and a utility function U, the problem of the economic agent is on the one hand to cover this contingent asset determining the optimal strategies $\hat{u}$ and $\hat{u}^0$ for investment in risky assets respectively in the presence of the option and in its absence. The economic agent sells the option at price x and buys this same option at price p^b from another agent to cover it. Therefore, this problem can be seen as a reinsurance problem. Market being viable but incomplete, for the determination of the optimal values, it is based on the maximization of the utility of the terminal wealth of its portfolio $X_T^{x,u}$. For the coverage of this contingent asset, it assumes that the payoff is rather $\Pi(S_T, B(z_T) + \bar{B})$ which depends on the price of exercise S_T of risky asset and subject to certain basic risks related to the market environment denoted $B(z_T)+\bar{B}$ which is an unobservable random variable due for example to transaction costs (cost of transport, of transit, etc.), inflation, natural disasters, etc. we propose to determine the optimal quantity to invest in the risky asset. We use the variable z_t to describe all the unobservable risks and uncertain related to the market environment at a date t, the value of the economic agent's portfolio is described with the variable $X_t^{x,u}$ under the strategy u_t and the initial wealth x with which he invests the quantity u_t of wealth in the asset risky and the remainder $X_t^{x,u}-u_t$in the semi-risk asset. The financial agent holds in his portfolio at time t in addition to the risky asset whose price is S_t, a semi-risky asset whose price S_t^0 is a deterministic function of a stochastic interest rate r_t.

The above problem was considered by the authors in [30], (see also [12], [6]). However, they were interested in classical utility indifference pricing and they also assume that the interest rate was deterministic. In this work, we assume that the interest rate is stochastic.

Given a finite horizon T, we consider a complete filtered probability space (Ω, F, $\{F_t\}_{0\in[0,T]}$; P). Let $(W_t^1, W_t^2)_{0\le t\le T}$, $(W_t^3)_{0\le t\le T}$ two Brownian motions. The financial market is made of two assets: one risky asset whose price is S_t and one semi-risky asset whose price is S_t^0 at time t with following dynamic:

$$dS_t^0 = r_t S_t^0 dt, \quad S_0^0 > 0 \qquad (1)$$

Where r_t is the interest rate.

As in [30], we assume that the dynamic price of the risky asset is given by a stochastic factor model. In particular, we assume that its drift depends on an unobservable factor Z_t. We assume that the drift of the risky asset μ also depends on interest rate r_t. Thus the dynamic of the risky asset is given by the following SDE:

$$dS_t = \mu(r_t, Z_t)\, S_t dt + \sigma_1 S_t dW_t^1, \quad S_0 = s > 0. \qquad (2)$$

Where μ is a deterministic function.

To remove the objection related to the possible negative nature of interest rate derived from the Gaussian model of Vasicek, Cox, Ingersoll and Ross (1985) introduced the so-called "square" root model, on the spot rate, in the risk-neutral universe. So in this work we let us retain the following CIR interest rate model:

$$dr_t = a(b-r_t)dt + \sigma_2\sqrt{r_t}dW_t^2, \quad r_0>0, \quad \sigma_2>0, \quad \sigma_2^2<2ab \qquad (3)$$

In this work, we are using CIR(Cox, Ingersoll and Ross) interest rate model.

Uncertain related events to the market environment can have a positive or a negative effect. So we assume that the unobservable external factor $\{Z_t,\ t\ge 0\}$ satisfy the following Ornstein-Uhlenbeck process given by SDE:

$$dZ_t = k(\beta - Z_t)dt + \sigma_3 dW_t^3,\ Z_0,\ \sigma_3 > 0 \qquad (4)$$

Let u_t be the amount of wealth invested in the risky asset at time t. Let $\{X_t^{x,u}, t\in[0,T]\}$ be the stochastic process modeling the evolution of portfolio wealth. We write$X_t^{x,u}$ to say the value of the wealth at time t, with initial value x under the strategy u. Then by the self-financing condition, $X_t^{x,u}$ is given by:

$$\begin{aligned} dX_t^{x,u} &= u_t\frac{dS_t}{S_t} + (X_t^{x,u}-u_t)\frac{dS_t^0}{S_t^0} \\ &= u_t\left[\mu(r_t,Z_t)dt + \sigma_1 dW_t^1\right] + (X_t^{x,u}-u_t)\,r_t dt, \quad X_0^{x,u}=x \end{aligned} \qquad (5)$$

To summarize, we have the following system:

$$\begin{cases} dS_t = S_t\mu(r_t,Z_t)dt + S_t\sigma_1 dW_t^1, & S_0 = s > 0 \\ dr_t = a(b-r_t)dt + \sigma_2\sqrt{r_t}dW_t^2 & r_0 > 0 \quad \sigma_2 > 0, \quad \sigma_2^2 < 2ab \\ dZ_t = k(\beta - Z_t)dt + \sigma_3 dW_t^3, & Z_0, \\ dS_t^0 = r_t S_t^0 dt, & S_0^0 > 0 \\ dX_t^{x,u} = \left[r_t X_t^{x,u} + u_t\left(\mu(r_t,Z_t) - r_t\right)\right]dt + u_t\sigma_1 dW_t^1, & X_0^{x,u} = x > 0 \end{cases} \tag{6}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 > 0$ are positive constancies. μ a deterministic function of r_t and Z_t. a, b, k, β the constancies.

Problem:

For the payoff $\Pi(S_T,B)$ of European option and a utility function U, the economic agent sells the option at the price x. He receives x and uses it to reinsure himself with another economic agent by buying the same contract at price p^b. While trying to maximize the expectation of the utility of the terminal wealth of his portfolio, his problem is: Determine the optimal management strategies $(\hat{u}_t)_{0\le t\le T}$, $(\hat{u}_t^0)_{0\le t\le T}$ representing respectively the quantities to invest in the risky asset in the presence of the option and in its absence, in the aim of maximizing the utility of the terminal wealth of its portfolio. That is to say the following mathematical problem: Determine the optimal

strategies $(\hat{u}_t)_{0\le t\le T}$ and $(\hat{u}_t^0)_{0\le t\le T}$, so that:

$$V_\Pi(x-p^b) = \sup_{u\in U_{ad}} \mathbb{E}[U(X_T^{x-p^b,u} + \Pi(S_T,B))] = \mathbb{E}[U(X_T^{x-p^b,\hat{u}} + \Pi(S_T,B))] \tag{7}$$

$$V_0(x) = \sup_{u\in U_{ad}} \mathbb{E}[U(X_T^{x,u})] = \mathbb{E}[U(X_T^{x,\hat{u}_0})] \tag{8}$$

with $U_{ad} \subset \mathbb{R}$ the set of admissible strategies u, $\mathfrak{F}^{(s,r)}$- progressively a.s measurable, contained in a bounded closed set. The optimal control $(\hat{u}_t)_{0\le t\le T}$ is the optimal strategy allowing the economic agent to cover the option with payoff $\Pi(S_T, B(z_T) + \bar{B})$ by maximizing the utility of the terminal wealth of its portfolio with the purchase of this option; $(\hat{u}_t^0)_{0\le t\le T}$ this optimal strategy in the absence of this contingent asset. The observable variables are S_t and r_t. The set of information at date t is the $\sigma-$ algebra $\mathrm{F}_t = \mathrm{F}^{(s,r)} = \sigma(S_{t1}, r_{t1}, 0 \le t_1 \le t)$ generated by all random variables S_{t1} and r_{t1}, $0 \le t_1 \le t$.

Consider: $Y_t^1 = \log S_t, Y_t^2 = r_t$, we have $S_t = \exp Y_t^1$, and (6) becomes:

$$\begin{cases} dY_t^1 = \left[\mu(r_t,Z_t) - \frac{1}{2}\sigma_1^2\right]dt + \sigma_1 dW_t^1, & Y_0^1 = log(s) \\ dY_t^2 = a(b-r_t)dt + \sigma_2\sqrt{r_t}dW_t^2, & Y_0^2 > 0 \quad \sigma_2^2 < 2ab \\ dZ_t = k\left(\beta - Z_t\right)dt + \rho_1\sigma_3 dW_t^1 + \rho_2\sigma_3 dW_t^2 + \sigma_3\sqrt{1-\rho_1^2}dW_t^{1^\perp} + \sigma_3\sqrt{1-\rho_2^2}dW_t^{2^\perp}, & Z_0 \\ dX_t^{x,u} = \left[r_t X_t^{x,u} + u_t\left(\mu(r_t,Z_t) - r_t\right)\right]dt + \sigma_1 u_t dW_t^1, & X_0^{x,u} = x > 0 \end{cases} \tag{9}$$

where $\rho_1 = corr(W^1,W^3)$, $\rho_2 = corr(W^2,W^3)$ are respectively the correlation coefficients between W^1 and W^3, W^2 and W^3. (W^1,W^2) and $(W^{1\perp},W^{2\perp})$ independent.

With assumption $B = B(z_T) + \bar{B}$, where B is a smooth function and $\bar{B}$ is a random variable independent of F_T. we can write (7) as follows:

$$\begin{aligned} V_\Pi(x) &= \sup_{u\in U_{ad}} \mathbb{E}\left[U(X_T^{x,u} + \Pi(S_T,B))\right] \qquad \text{by independence of } \bar{B} \text{ and } Z_T. \quad (10) \\ &= \sup_{u\in U_{ad}} \mathbb{E}\left[\int_{\mathbb{R}} U\left(X_T^{x,u} + \Pi(S_T, B(Z_T)+\bar{b})\right) d\mathbb{P}_{\bar{B}}\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} U\left(X_T^{x,\hat{u}} + \Pi(S_T, B(Z_T)+\bar{b})\right) d\mathbb{P}_{\bar{B}}\right] \end{aligned}$$

3. From partial to complete information

We go as in [2], [6], [12], [24], [30] using filtering theory with the Zakai equations to transform the control problem (9) - (10) to partial information into complete information problem. Unlike in [24] where one replaces only Z_t with its conditional expectation knowing $F^{Y}{}_{T}$, Here, we are going to use as in [6], [30], [12], [2] the property of iterated conditional expectation and replace any function $f(Z_T)$with its conditional expectation knowing $F^{Y}{}_{T}$, but here with two observable variables (S_t and r_t). Let us summarize a brief general result presented in [6] and [30].

Given two independent Brownian movements W and $W^{\perp}$ respectively p and $q-$ dimensional, $(Y_t)_{0\leq t\leq T}$ the observable variable $n-$dimensional, $(Z_t)_{0\leq t\leq T}$ the variable not observable $m-$dimensional with dynamics:

$$\begin{cases} dY_t = h(t,Z_t,Y_t)dt + \sigma(t,Y_t)dW_t, & Y_0 \\ dZ_t = g(t,Z_t,Y_t)dt + \alpha(t,Y_t,Z_t)dW_t + \gamma(t,Y_t,Z_t)dW_t^{\perp}, & Z_0 = \varepsilon \end{cases} \tag{11}$$

We assume the following:

i) $h(t,z,y) : [0;T] \times R^m \times R^n \longrightarrow R^n$ is globally continuous and of linear growth (in z and y:

$|h(t,z,y)| \leq k(1+ | z |)+ | y |)$.

ii) $g(t,z,y) : [0;T] \times R^m \times R^n \longrightarrow R^m$ is uniformly continuous in z, y is bounded and twice continuously differentiable with bounded derivatives.

iii) $\sigma(t,y) : [0;T] \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n,\mathbb{R}^p)$ is uniformly continuous, bounded, three times continuously differentiable with bounded derivative, satisfies the following:$\sigma\sigma' \geq \lambda I$ for all y and t, for some constant $\lambda > 0$ (uniform ellipticity condition). [1](uniform ellipticity condition).

iv) $\alpha(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^m,\mathbb{R}^d), \gamma(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^m,\mathbb{R}^q)$ are uniformly continuous and α is uniformly elliptic.

v) h,σ,g and γ are globally lipschitz of y and z.

<u>Remark </u>3.1.
As mentioned in [6], [30], these general results can well be applied to our initial model although the drifts are not bounded. A standard localization of the argument can be used to have a linearly increasing drift.

Let $D_t = D(t,Y_t) = (\sigma\sigma)(t,Y_t)$ which we assume to be symmetrical and invertible. $\phi(t)$ defined by :

$$d\varphi_t = -\varphi_t h^T(t,Z_t,Y_t)D_t^{-\frac{1}{2}}(t,Y_t)dW_t, \qquad \varphi_0 = 1$$

ϕ is a supermartingale with $E[\phi_t] = 1$ $\forall t \in [0,T]$. Therefore by Girsanov's theorem, we define a new probability measure $\tilde{P}$ such that $\forall t \in [0,T]$

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathfrak{F}_t} := \varphi_t \quad (\ d\tilde{\mathbb{P}} = \varphi_t d\mathbb{P} \quad \text{on } F_t, \forall t \in [0,T]).$$

and there exists a Brownian motion $\tilde{W}$ under $\tilde{P}$ such that: $dY_t = \sigma(t,Y_t)d\tilde{W}_t$

$$\begin{aligned} dZ_t &= \left[g(t,Z_t,Y_t) - \alpha^T(t,Z_t,Y_t)h^T(t,Z_t,Y_t)D_t^{-\frac{1}{2}}\right]dt + \\ &\quad + \alpha^T(t,Z_t,Y_t)D_t^{-\frac{1}{2}}dY_t + \gamma(t,Z_t,Y_t)dW_t^{\perp}. \end{aligned}$$

; Let $(Y_t, t \in [0,T])$ be the process defined by:

$$d\tilde{Y}_t = D^{-\frac{1}{2}}(t)dY_t. \tag{12}$$

Then $\tilde{Y}$ is a Brownian motion under P and $\tilde{Y}$and $W^{\perp}$ are two independent Brownian motions(see[2]). In addition, using the fact that D_t is invertible, we have that: $F_t^Y = \mathfrak{F}_t^{\tilde{Y}}$. Let us set

$$
\begin{aligned}
K_t =: \frac{1}{\rho_t} &= \exp\left\{\int_0^t h^\top(s,Z_s,Y_s)D^{-\frac{1}{2}}(s)dW_s \right.\\
&\quad \left. + \frac{1}{2}\int_0^t h^\top(s,Z_s,Y_s)D^{-1}(s)h(s,Z_s,Y_s)d_s\right\}\\
&= \exp\left\{\int_0^t h^\top(s,Z_s,Y_s)D^{-1}(s)dY_s \right.\\
&\quad \left. + \frac{1}{2}\int_0^t h^\top(s,Z_s,Y_s)D^{-1}(s)h(s,Z_s,Y_s)d_s\right\}
\end{aligned}
$$

Then K_t is a martingale.
Let

$$\phi = (\phi(t,z,w),(t,z,w) \in [0,T]\times\mathbb{R}^d\times\Omega),$$

be a process such that for all $f \in C_0^\infty(\mathbb{R}^d)$ [3]. Where [3] $C_0^\infty(\mathbb{R}^d)$=space of functions C^∞ on $\mathbb{R}^d$ with compact support.

We have:

$$\tilde{E}\left[f(Z_t)K_t|\mathfrak{F}_t^Y\right] = \int_{\mathbb{R}^d} f(z)\phi(t,z)dz, \tag{13}$$

$\tilde{E}$ its expectation under $\tilde{P}$. Then $\phi(t,z)$ is called the unormalized conditional density of Z_t given $\mathfrak{F}_t^Y$.

Applying the theorem of Îto to ($K_t f(Z_t)$)and using the integration by parts, ϕ satisfy the following Zakai equation (see [6], [30], [2]):

$$
\begin{aligned}
d\phi(t,z) = &\left[\frac{1}{2}\sum_{i,j}\sum\frac{\partial^2}{\partial z_i\partial z_j}\left\{\left[\gamma\gamma^t+\alpha\alpha^t\right]_{i,j}\phi(t,z)\right\} - \sum_i\frac{\partial g_i\phi(t,z)}{\partial z_i}\right]dt+\\
&\left[h-\sum_i\frac{\partial}{\partial z_i}(\alpha^i\phi(t,z))\right]d\tilde{Y}_t; \qquad \phi(0,z)=\xi(z)\\
=&L_Z^*\phi(t,z)dt + M^*\phi(t,z)d\tilde{Y}_t; \qquad \phi(0,z)=\xi(z)
\end{aligned} \tag{14}
$$

where $\xi(z)$ is the density of Z_0

$$
\begin{aligned}
L^*\phi(t,z) &= \frac{1}{2}\sum_{i,j}\sum\frac{\partial^2}{\partial z_i\partial z_j}\left\{\left[\gamma\gamma^t+\alpha\alpha^t\right]_{i,j}\phi(t,z)\right\} - \sum_i\frac{\partial\left(g_i\phi(t,z)\right)}{\partial z_i}\\
&= \mathfrak{L}\phi(t,z) + i\phi(t,z)\\
M^*\phi(t,z) &= h - \sum_i\frac{\partial}{\partial x_i}(\alpha^i\phi(t,z))d\tilde{Y}(t)
\end{aligned}
$$

By combining (9), (10), (13) and (14) we transform the control problem under partial observation into a problem with SDE into a control problem (9) - (10) at full information with PSDE:

$$
\begin{aligned}
V_\Pi(x) &= \sup_{u\in U_{ad}}\tilde{\mathbb{E}}\left[K_T\int_{\mathbb{R}}U(X_T^{x,u}+\Pi(S_T,B(Z_T)+\bar{b}))d\mathbb{P}_{\bar{B}}\right]\\
&= \sup_{u\in U_{ad}}\tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[K_T\int_{\mathbb{R}}U(X_T^{x,u}+\Pi(S_T,B(Z_T)+\bar{b}))d\mathbb{P}_{\bar{B}}|\mathfrak{F}_T^{Y_T}\right]\right]\\
V_\Pi(x) &= \sup_{u\in U_{ad}}\tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}U(X_T^{x,u}+\Pi(S_T,B(z)+\bar{b})\phi(T,z))d\mathbb{P}_{\bar{B}}dz\right]\\
&= \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}U(X_T^{x,\hat{u}}+\Pi(S_T,B(z)+\bar{b}))\phi(T,z))d\mathbb{P}_{\bar{B}}dz\right]
\end{aligned} \tag{15}
$$

With $\varphi(t,z)$ according to (14) solution of the stochastic PDE:

$$
\begin{cases}
d\phi(t,z) = & \left[\sigma_3^2 \frac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right] dt \\
& + \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right] d\tilde{Y}_t^1 + \left[a(b-r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right] d\tilde{Y}_t^2 \\
& = L_Z^*\phi(t,z)dt + M_1^*\phi(t,z)d\tilde{Y}_t^1 + M_2^*\phi(t,z)d\tilde{Y}_t^2 \\
\phi(0,z) & = \xi(z)
\end{cases}
\tag{16}
$$

With μ a function such as the last equation of the system which is an SPDE admits a unique strong solution. let's remind that $Y_t = logS_t$, we have $S_t = \exp Y_t$, from (9) the fact that :

$$
\begin{cases}
dS_t = \frac{\sigma_1^2 S_t}{2}dt + S_t dY_t^1, & S_0 = s \\
dr_t = dY_t^2, & r_0 \\
dX_t^{x,u} = \left[r_t X_t - u_t(r_t - \frac{1}{2}\sigma_1^2)\right] dt + u_t dY_t^1, & X_0^{x,u} = x.
\end{cases}
\tag{17}
$$

Finally by combining (17), (16), (9) we have:

$$
\begin{cases}
dS_t = \frac{\sigma_1^2 S_t}{2}dt + \sigma_1 S_t d\tilde{Y}_t^1, & S_0 = s \\
dr_t = \sigma_2\sqrt{r_t}d\tilde{Y}_t^2, & r_0, \\
dX_t^{x,u} = \left[r_t X_t^{x,u} - u_t(r_t - \frac{1}{2}\sigma_1^2)\right] dt + u_t\sigma_1 d\tilde{Y}_t^1, & X_0^{x,u} = x \\
d\phi(t,z) = \left[\sigma_3^2\dfrac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\dfrac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right] dt & \\
\qquad + \left[\mu(r_t,z) - \dfrac{1}{2}\sigma_1^2 - \rho_1\sigma_3\dfrac{\partial\phi}{\partial z}(t,z)\right] d\tilde{Y}_t^1 + \left[a(b-r_t) - \sigma_3\rho_2\dfrac{\partial\phi}{\partial z}(t,z)\right] d\tilde{Y}_t^2 & \phi(0,z) = \xi(z) \\
\qquad = L_Z^*\phi(t,z)dt + M_1^*\phi(t,z)d\tilde{Y}_t^1 + M_2^*\phi(t,z)d\tilde{Y}_t^2 &
\end{cases}
.
\tag{18}
$$

4. A sufficient stochastic maximum principle

We will establish in this section, a stochastic maximum principle that we will use in the next section to determine the optimal amount to invest in the risky asset, the optimal value of the portfolio at each instant. Unlike as in [6, 30], we will use two observable variables.

Let T be a fixed exercise date, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\in[0,T]}; \mathbb{P})$. a filtered probability space on which we have two Brownian motions W^1 and W^2. We consider the controlled diffusion below which describes the dynamics of the various state processes:

$$
\begin{cases}
dY_t^1 = b_1(t,Y_t^1,u_t)dt + \sigma_{11}(t,Y_t^1,u_t)dW_t^1 + \sigma_{12}(t,Y_t^1,u_t)dW_t^2, & Y_0^1 = y_0^1 \\
dY_t^2 = b_2(t,Y_t^2,u_t)dt + \sigma_{21}(t,Y_t^2,u_t)dW_t^1 + \sigma_{22}(t,Y_t^2,u_t)dW_t^2, & Y_0^2 = y_0^2 \\
dX_t = b_3(t,X_t,Y_t^1,Y_t^2,u_t)dt + \sigma_{31}(t,X_t,u_t)dW_t^1 + \sigma_{32}(t,X_t,u_t)dW_t^2, & X_0 = x \\
d\phi(t,z) = [L\phi(t,z) + b_4(t,z,\phi(t,z),\dfrac{\partial\phi}{\partial z}(t,z)),u_t]dt + \sigma_{41}(t,z,\phi(t,z),\dfrac{\partial\phi}{\partial z}(t,z),u_t)dW_t^1 & \\
\qquad + \sigma_{42}(t,z,\phi(t,z),\dfrac{\partial\phi}{\partial z}(t,z),u_t)dW_t^2, \quad \phi(0,z) = \xi(z), \ z\in\mathbb{R} & \\
\qquad \lim\limits_{\|z\|\to+\infty}\phi(t,z) = 0 \quad \forall t\in[0,T] &
\end{cases}
\tag{19}
$$

where L is a linear differential operator. b_1, b_2, b_3, b_4, σ_{11}, σ_{12}, σ_{21}, σ_{22}, σ_{31}, σ_{32}, σ_{41}, σ_{42} the given functions satisfying the conditions of existence and uniqueness of strong solutions of the above system, and L^* the formal adjoint of L.

Let f and g be given functions C^1 in their arguments. We consider the objective function:

$$
\begin{aligned}
J(u) = & \mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\bar{b},u_t)dzd\mathbb{P}_{\bar{B}}dt\right. \\
& \left. + \int_{\mathbb{R}}\int_{\mathbb{R}} g(z,Y_T^1,Y_T^2,X_T,\phi(T,z),\bar{b},u_T)dzd\mathbb{P}_{\mathbb{B}}\right]
\end{aligned}
\tag{20}
$$

We note U_{ad} the set of admissible controls contained in the set of controls F_t- predictable such that the above system has a single strong solution and

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \left|f(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\bar{b},u_t)\right| dz d\mathbb{P}_{\bar{B}} dt + \int_{\mathbb{R}} \int_{\mathbb{R}} \left|g(z,Y_T^1,Y_T^2,X_T,\phi(T,z),\bar{b},u_T)\right| dz d\mathbb{P}_{\mathbb{B}}\right] < \infty.$$

Problem 1. Determine the value function

$$J(\hat{u}) = \sup_{u \in U_{ad}} J(u) \tag{21}$$

under conditions (19).

That is to say to seek the optimal control $\hat{u} \in U_{ad}$ which maximizes the objective function J
As such, we define the associated Hamiltonian by:

$$H : [0,T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$$

$$H(t,z,y_1,y_2,x,\phi,\phi',u,p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4,q_1',q_2',q_3',q_4') = H(t,z,y,x,\phi,\phi',u,p,q,q')$$
. with $y = (y_1,y_2), p = (p_1,p_2,p_3,p_4), q = (q_1,q_2,q_3,q_4), q' = (q_1',q_2',q_3',q_4')$.

$$\begin{aligned} H(t,z,y_1,y_2,x,\phi,\phi',u,p,q,q') = &\int_{\mathbb{R}} f(t,z,y_1,y_2,x,\phi,\bar{b},u) d\mathbb{P}_{\bar{B}} + b_1(t,y_1,u)p_1 + b_2(t,y_2,u)p_2 \\ &+ b_3(t,y_1,y_2,x,u)p_3 + b_4(t,z,\phi,\phi',u)p_4 + \sigma_{11}(t,y_1,u)q_1 + \sigma_{21}(t,y_2,u)q_2 \\ &+ \sigma_{31}(t,x,u)q_3 + \sigma_{41}(t,z,\phi,\phi',u)q_4 + \sigma_{12}(t,y_1,u)q_1' \\ &+ \sigma_{22}(t,y_2,u)q_2' + \sigma_{32}(t,x,u)q_3' + \sigma_{42}(t,z,\phi,\phi',u)q_4' \end{aligned} \tag{22}$$

Where $\phi' = \frac{\partial \phi}{\partial z}$. We suppose that H is differentiable in the variables y_1, y_2, x, ϕ and ϕ'.

For $u \in U_{ad}$, we consider the adjoint processes satisfying the backward stochastic differential equations in the unknown $p_1(t,z), q_1(t,z), q_1'(t,z), p_2(t,z), q_2(t,z), q_2'(t,z), p_3(t,z), q_3(t,z), q_3'(t,z)$, $p_4(t,z), q_4(t,z), q_4'(t,z) \in \mathbb{R}$ with the system of adjoint equations:

$$\begin{cases} -dp_1 = \frac{\partial H}{\partial y_1}(t,z)dt - q_1 dW_t^1 - q_1' dW_t^2, & p_1(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial y_1}(z,\bar{b}) d\mathbb{P}_{\bar{B}} \\ -dp_2 = \frac{\partial H}{\partial y_2}(t,z)dt - q_2 dW_t^1 - q_2' dW_t^2, & p_2(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial y_2}(z,\bar{b}) d\mathbb{P}_{\bar{B}} \\ -dp_3 = \frac{\partial H}{\partial x}(t,z)dt - q_3 dW_t^1 - q_3' dW_t^2, & p_3(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial x}(z,\bar{b}) d\mathbb{P}_{\bar{B}} \\ -dp_4 = \left[\frac{\partial H}{\partial \phi}(t,z) + L^* p_4(t,z) - \frac{\partial}{\partial z}\left(\frac{\partial H(t,z)}{\partial \phi'}\right)\right] dt - q_4 dW_t^1 - q_4' dW_t^2, & p_4(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial \phi}(z,\bar{b}) d\mathbb{P}_{\bar{B}} \\ \quad , \quad \lim_{\|x\| \to +\infty} p_4(t,z) = 0 & \end{cases} \tag{23}$$

With short notations: $g(z,\bar{b}) = g(z,Y_T^1,Y_T^2,X_T,\varphi(T,z),\bar{b},u_T)$ and
$H(t,z) = H(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\phi'(t,z),u_t,p(t,z),q(t,z),q\ (t,z))$

<u>Remark</u> 4.1.

If we suppose for example that the coefficients of the controlled diffusion, and our functions are fairly regular then there is existence and uniqueness of classical strong solutions of our backward stochastic differential equations as well as of the backward stochastic partial differential equation constituting the process of associated adjunct equations.

We have the following theorem:

Theorem 4.1. *(A Stochastic Maximum Principle)*
Let $\hat{u} \in U_{ad}$ *with the corresponding solutions* $\hat{Y}_t^1, \hat{Y}_t^2, \hat{X}_t, \hat{\phi}(t,z)$ *of*(19); $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4), \hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4), \hat{q'} = (\hat{q'_1}, \hat{q'_2}, \hat{q'_3}, \hat{q'_4})$ *of* (21) *and* (23)
Let the following conditions be satisfied:

(1) *The function* $g : (y_1, y_2, x, \phi, u) \longmapsto g(z, y_1, y_2, x, \phi, \bar{b}, u)$ *is concave in* y_1, y_2, x *and* ϕ *for all* $z \in \mathbb{R}, \bar{b} \in \mathbb{R}, u \in U_{ad}$

(2) $H(t, z, y_1, y_2, x, \phi, \phi', \hat{u}, \hat{p}, \hat{q}, \hat{q'}) = \sup_{u \in U_{ad}} H(t, z, y_1, y_2, x, \phi, \phi', u, \hat{p}, \hat{q}, \hat{q'})$ *exists* $\forall y_1, y_2, x, \phi, \phi'$

(3) *The function* $h : (y_1, y_2, x, \phi, \phi') \longmapsto H(t, z, y_1, y_2, x, \phi, \phi', \hat{u}, \hat{p}, \hat{q}, \hat{q'})$ *is concave in* y_1, y_2, x, ϕ *and* ϕ' *for all* $(t,z) \in [0,T] \times \mathbb{R}$

(4) *With the integrability conditions below satisfied:*

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left\{ (\sigma_{11} - \hat{\sigma}_{11})^2 \hat{p}_1^2(t,z) + (\sigma_{12} - \hat{\sigma}_{12})^2 \hat{p}_1^2(t,z) + \left(Y_t^1 - \hat{Y}_t^1\right)^2 \hat{q}_1^2(t,z) + \left(Y_t^1 - \hat{Y}_t^1\right)^2 \hat{q'}_1^2(t,z) + (\sigma_{21} - \hat{\sigma}_{21})^2 \hat{p}_2^2(t,z) + (\sigma_{22} - \hat{\sigma}_{22})^2 \hat{p}_2^2(t,z) + \left(Y_t^2 - \hat{Y}_t^2\right)^2 \hat{q}_2^2(t,z) + \left(Y_t^2 - \hat{Y}_t^2\right)^2 \hat{q'}_2^2(t,z) + (\sigma_{31} - \hat{\sigma}_{31})^2 \hat{p}_3^2(t,z) + (\sigma_{32} - \hat{\sigma}_{32})^2 \hat{p}_3^2(t,z) + \left(X_t - \hat{X}_t\right)^2 \hat{q}_3^2(t,z) + \left(X_t - \hat{X}_t\right)^2 \hat{q'}_3^2(t,z) \right\} dtdz \right] < \infty. \text{ and}$$

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left\{ (\sigma_{41} - \hat{\sigma}_{41})^2 \hat{p}_4^2(t,z) + (\sigma_{42} - \hat{\sigma}_{42})^2 \hat{p}_4^2(t,z) + \left(\phi(t,z) - \hat{\phi}_1(t,z)\right)^2 \hat{q}_4^2(t,z) + \left(\phi(t,z) - \hat{\phi}_1(t,z)\right)^2 \hat{q'}_4^2(t,z) \right\} dtdz \right] < \infty.$$

Then the control $\hat{u}$ *is optimal for the problem* (21).

Proof. .
. Let us show that $J(\hat{u}) \geq J(u) \ \ \forall u \in U_{ad}$.
Recall that

$$\begin{aligned} H(t, z, y_1, y_2, x, \phi, \phi', u, p, q, q') = & \int_{\mathbb{R}} f(t, z, y_1, y_2, x, \phi, \bar{b}, u) d\mathbb{P}_{\bar{B}} + b_1(t, y_1, u)p_1 + b_2(t, y_2, u)p_2 \\ & + b_3(t, y_1, y_2, x, u)p_3 + b_4(t, z, \phi, \phi', u)p_4 + \sigma_{11}(t, y_1, u)q_1 + \sigma_{21}(t, y_2, u)q_2 \\ & + \sigma_{31}(t, x, u)q_3 + \sigma_{41}(t, z, \phi, \phi', u)q_4 + \sigma_{12}(t, y_1, u)q_1' \\ & + \sigma_{22}(t, y_2, u)q_2' + \sigma_{32}(t, x, u)q_3' + \sigma_{42}(t, z, \phi, \phi', u)q_4'. \end{aligned}$$

Let's pose

$H(t,z) = H(t, z, Y_t^1, Y_t^2, X_t, u_t, \phi(t,z), \phi'(t,z), \hat{p}, \hat{q}, \hat{q'})$,
$\hat{H}(t,z) = H(t, z, \hat{Y}_t^1, \hat{Y}_t^2, \hat{X}^t, \hat{u}_t, \hat{\phi}(t,z), \hat{\phi}'(t,z), \hat{p}, \hat{q}, \hat{q'})$,
$f(t,z,\bar{b}) = f(t, z, Y_t^1, Y_t^2, X_t, \phi(t,z), \bar{b}, u_t)$, $\hat{f}(t,z,\bar{b}) = f(t, z, \hat{Y}_t^1, \hat{Y}_t^2, \hat{X}_t, \hat{\phi}(t,z), \bar{b}, \hat{u})$,
$g(z,\bar{b}) = g(z, Y_T^1, Y_T^2, X_T, \phi(T,z), \bar{b}, u)$, $\hat{g}(z,\bar{b}) = g(z, \hat{X}_T, \hat{Y}_T^1, \hat{Y}_T^2, \hat{\phi}(T,z), \bar{b}, u)$,

$$b_1(t)=b_1(t,Y_t^1,u_t),\hat{b}_1(t)=\hat{b}_1(t,\hat{Y}_t^1,\hat{u}_t),\ \ b_2(t)=b_2(t,Y_t^1,u_t),\hat{b}_2(t)=\hat{b}_2(t,\hat{Y}_t^1,\hat{u}_t)$$
$$b_3(t)=b_1(t,X_t,Y_t^1,Y_t^2,u_t),\ \ \hat{b}_3(t)=\hat{b}_3(t,\hat{X}_t,\hat{Y}_t^1,\hat{Y}_t^2,\hat{u}_t),$$
$$b_4(t,z)=b_4(t,z,\phi(t,z),\phi'(t,z)),u_t),\ \ \hat{b}_4(t,z)=\hat{b}_4(t,z,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{u}(t)),$$
$$\sigma_{11}(t)=\sigma_{11}(t,Y_t^1,u_t),\hat{\sigma}_{11}(t)=\hat{\sigma}_{11}(t,\hat{Y}_t^1,\hat{u}_t),\ \ \sigma_{12}(t)=\sigma_{12}(t,Y_t^1,u_t),\hat{\sigma}_{12}(t)=\hat{\sigma}_{12}(t,\hat{Y}_t^1,\hat{u}_t),$$

$$\sigma_{21}(t)=\sigma_{21}(t,Y_t^2,u_t),\hat{\sigma}_{21}(t)=\hat{\sigma}_{21}(t,\hat{Y}_t^2,\hat{u}_t),\ \sigma_{22}(t)=\sigma_{22}(t,Y_t^2,u_t),\hat{\sigma}_{22}(t)=\hat{\sigma}_{22}(t,\hat{Y}_t^2,\hat{u}_t)$$
$$\sigma_{31}(t)=\sigma_{31}(t,X_t,u_t),\hat{\sigma}_{31}(t)=\hat{\sigma}_{31}(t,\hat{X}_t,\hat{u}_t),\ \ \sigma_{32}(t)=\sigma_{32}(t,X_t,u_t),\hat{\sigma}_{32}(t)=\hat{\sigma}_{32}(t,\hat{X}_t,\hat{u}_t),$$
$$\sigma_{41}(t,z)=\sigma_{41}(t,z,\phi(t,z),\phi'(t,z),u_t),\ \ \hat{\sigma}_{41}(t,z)=\hat{\sigma}_{41}(t,z,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{u}_t),$$
$$\sigma_{42}(t,z)=\sigma_{42}(t,z,\phi(t,z),\phi'(t,z),u_t),\ \ \hat{\sigma}_{42}(t)=\hat{\sigma}_{42}(t,z,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{u}_t)$$

$\hat{p}_1=\hat{p}_1(t,z),\hat{p}_2=\hat{p}_2(t,z),\hat{p}_3=\hat{p}_3(t,z),\hat{p}_4=\hat{p}_4(t,z),\hat{q}_1=\hat{q}_1(t,z),\hat{q}_2=\hat{q}_2(t,z),\hat{q}_3=\hat{q}_3(t,z),\hat{q}_4=\hat{q}_4(t,z),\hat{q'}_1=\hat{q'}_1(t,z),\hat{q'}_2=\hat{q'}_2(t,z),\hat{q'}_3=\hat{q'}_3(t,z),\hat{q'}_4=\hat{q'}_4(t,z)$, we have :

$$\begin{aligned}\int_{\mathbb{R}} f(t,z,\bar{b})d\mathbb{P}_{\bar{B}} &= H(t,z)-b_1(t)\hat{p}_1-b_2(t)\hat{p}_2-b_3(t)\hat{p}_3-b_4(t)\hat{p}_4-\sigma_{11}\hat{q}_1\\ &\quad -\sigma_{21}\hat{q}_2-\sigma_{31}\hat{q}_3-\sigma_{41}\hat{q}_4-\sigma_{12}\hat{q'}_1-\sigma_{22}\hat{q'}_2-\sigma_{32}\hat{q}'_3+\sigma_{42}\hat{q}'_4\end{aligned} \tag{24}$$

$$\begin{aligned}\int_{\mathbb{R}} \hat{f}(t,z,\bar{b})d\mathbb{P}_{\bar{B}} &= \hat{H}(t,z)-\hat{b}_1(t)\hat{p}_1-\hat{b}_2(t)\hat{p}_2-\hat{b}_3(t)\hat{p}_3-\hat{b}_4(t)\hat{p}_4-\hat{\sigma}_{11}\hat{q}_1\\ &\quad -\hat{\sigma}_{21}\hat{q}_2-\hat{\sigma}_{31}\hat{q}_3-\hat{\sigma}_{41}\hat{q}_4-\hat{\sigma}_{12}\hat{q'}_1-\hat{\sigma}_{22}\hat{q'}_2-\hat{\sigma}_{32}\hat{q}'_3+\hat{\sigma}_{42}\hat{q}'_4\end{aligned} \tag{25}$$

$$\begin{aligned}J(\hat{u})-J(u) &= \mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\hat{f}(t,z,\bar{b})-f(t,z,\bar{b})\big)dtdzd\mathbb{P}_{\mathbb{B}}+\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\hat{g}(z,\bar{b})-g(z,\bar{b})\big)dzd\mathbb{P}_{\mathbb{B}}\right]\\ &= I_1+I_2\end{aligned} \tag{26}$$

From (24), (25) and (26) we have:

$$\begin{aligned}I_1 =&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\hat{f}(t,z,\bar{b})-f(t,z,\bar{b})\big)dtdzd\mathbb{P}_{\mathbb{B}}\right]\\ =&\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{b}_1(t)-b_1(t)\big)\hat{p}_1(t,z)-\big(\hat{b}_2(t)-b_2(t)\big)\hat{p}_2(t,z)-\big(\hat{b}_3(t)-b_3(t)\big)\hat{p}_3(t,z)\\ &-\big(\hat{b}_4(t,z)-b_4(t,z)\big)\hat{p}_4(t,z)-\big(\hat{\sigma}_{11}(t)-\sigma_{11}(t)\big)\hat{q}_1(t,z)-\big(\hat{\sigma}_{21}(t)-\sigma_{21}(t)\big)\hat{q}_2(t,z)\\ &-\big(\hat{\sigma}_{31}(t)-\sigma_{31}(t)\big)\hat{q}_3(t,z)-\big(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z)\big)\hat{q}_4(t,z)-\big(\hat{\sigma}_{12}(t)-\sigma_{12}(t)\big)\hat{q'}_1(t,z)\\ &-\big(\hat{\sigma}_{22}(t)-\sigma_{22}(t)\big)\hat{q'}_2(t,z)-\big(\hat{\sigma}_{32}(t)-\sigma_{32}(t)\big)\hat{q'}_3(t,z)-\big(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z)\big)\hat{q'}_4(t,z)\Big\}dtdz\Big]\end{aligned} \tag{27}$$

$$\begin{aligned}I_2 =&\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\{\hat{g}(z,\bar{b})-g(z,\bar{b})\}dzd\mathbb{P}_{\bar{\mathbb{B}}}\right]\\ \geq&\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}\Big\{\frac{\partial\hat{g}}{\partial x}(z,\bar{b})(\hat{X}_T-X_T)+\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})(\hat{Y}_T^1-Y_T^1)+\frac{\partial\hat{g}}{\partial y_2}(z,\bar{b})(\hat{Y}_T^2-Y_T^2)\\ &+\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})\big(\hat{\phi}(T,z)-\phi(T,z)\big)\Big\}dzd\mathbb{P}_{\mathbb{B}}\Big]\ \text{according to the concavity of } g \text{ in } \ \hat{Y}^1,\hat{Y}^2,\hat{X},\text{and } \hat{\phi}\\ \\ =&\mathbb{E}\Big[\int_{\mathbb{R}}\Big\{(\hat{X}_T-X_T)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial x}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}+(\hat{Y}_T^1-Y_T^1)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}+(\hat{Y}_T^2-Y_T^2)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_2}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}\\ &+\big(\hat{\phi}(T,z)-\phi(T,z)\big)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}\Big\}dz\Big]\qquad \text{since } Y^1,Y^2\ X,\text{and } \phi \ \text{ thus depend on } \ \bar{b}\end{aligned}$$

$$=\mathbb{E}\left[\int_{\mathbb{R}}\left\{(\hat{Y}_T^1-Y_T^1)\hat{p}_1(T,z)+(\hat{Y}_T^2-Y_T^2)\hat{p}_2(T,z)+(\hat{X}_T-X_T)\hat{p}_3(T,z)+(\hat{\phi}(T,z)-\phi(T,z))\hat{p}_4(T,z)\right\}dz\right]$$

hence from(23) $\hat{p}_1(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\bar{B}},\hat{p}_2(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_2}(z,\bar{B})d\mathbb{P}_{\bar{B}}$

$\hat{p}_3(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial x}(z,\bar{b})d\mathbb{P}_{\bar{B}}$ and $\hat{p}_4(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})d\mathbb{P}_{\bar{B}},$

hence from (23) $\hat{p}_1(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\bar{B}},\hat{p}_2(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_2}(z,\bar{B})d\mathbb{P}_{\bar{B}}$

$\hat{p}_3(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial x}(z,\bar{b})d\mathbb{P}_{\bar{B}}$ and $\hat{p}_4(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})d\mathbb{P}_{\bar{B}}$,

$$\begin{aligned}
=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\Big\{d(\hat{Y}_t^1-Y_t^1)\hat{p}_1(t,z)+(\hat{Y}_t^1-Y_t^1)d\hat{p}_1(t,z)+d<\hat{Y}_t^1-Y_t^1,\hat{p}_1(t,z)>+d(\hat{Y}_t^2-Y_t^2)\hat{p}_2(t,z)\\
&+(\hat{Y}_t^2-Y_t^2)d\hat{p}_2(t,z)+d<\hat{Y}_t^2-Y_t^2,\hat{p}_2(t,z)>+d(\hat{X}_t-X_t)\hat{p}_3(t,z)+(\hat{X}_t-X_t)d\hat{p}_3(t,z)\\
&+d<\hat{X}_t-X_t,\hat{p}_3(t,z)>+d(\hat{\phi}(t,z)-\phi(t,z))\hat{p}_4(t,z)+(\hat{\phi}(t,z)-\phi(t,z))d\hat{p}_4(t,z)\\
&+d<\hat{\phi}(t,z)-\phi(t,z),\hat{p}_4(t,z)>\Big\}dzdt\Big]\\
=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\Big\{\Big((\hat{b}_1(t)-b_1(t))dt+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))dW_1(t)+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))dW_t^2\Big)\hat{p}_1(t,z)\\
&+\Big((\hat{b}_2(t)-b_2(t))dt+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))dW_t^1+(\hat{\sigma}_{22}(t)-\sigma_{22}(t))dW_t^2\Big)\hat{p}_2(t,z)\\
&+\Big((\hat{b}_3(t)-b_3(t))dt+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))dW_t^1+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))dW_t^2\Big)\hat{p}_3(t,z)\\
&+\Big((L(\hat{\phi}(t,z)-\phi(t,z))dt+(\hat{b}_4(t,z)-b_4(t,z))dt+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))dW_t^1\\
&+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))dW_t^2)\hat{p}_4(t,z)+(\hat{Y}_t^1-Y_t^1)(-\frac{\partial\hat{H}}{\partial y_1}(t,z)dt+\hat{q}_1(t,z)dW_t^1+\hat{q}_1'(t,z)dW_t^2)\\
&+(\hat{Y}_t^2-Y_t^2)(-\frac{\partial\hat{H}}{\partial y_2}(t,z)dt+\hat{q}_2(t,z)dW_t^1+\hat{q}_2'(t,z)dW_t^2)\\
&+(\hat{X}_t-X_t)(-\frac{\partial\hat{H}}{\partial x}(t,z)dt+\hat{q}_3(t,z)dW_t^1+\hat{q}_3'(t,z)dW_t^2)\\
&+(\hat{\phi}(t,z)-\phi(t,z))\Big((-\frac{\partial\hat{H}}{\partial\phi}(t,z)-L^*\hat{p}_4(t,z)+\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)dt+\hat{q}_4(t,z)dW_t^1+\hat{q}_4'(t,z)dW_t^2\Big)\\
&+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))\hat{q}_1(t,z)dt+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))\hat{q}_1'(t,z)dt+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))\hat{q}_2(t,z)dt\\
&+(\hat{\sigma}_{22}(t)-\sigma_{22}(t))\hat{q}_2'(t,z)dt+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))\hat{q}_3(t,z)dt+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))\hat{q}_3'(t,z)dt\\
&+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))\hat{q}_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))\hat{q}_4'(t,z)\Big\}dz\Big]\\
=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\Big\{\Big(\hat{b}_1(t)-b_1(t)\Big)\hat{p}_1(t,z)+\Big(\hat{b}_2(t)-b_2(t)\Big)\hat{p}_2(t,z)+\Big(\hat{b}_3(t)-b_3(t)\Big)\hat{p}_3(t,z)dt\\
&+\Big(\hat{b}_4(t)-b_4(t)\Big)\hat{p}_4(t,z)+L(\hat{\phi}(t,z)-\phi(t,z))\hat{p}_4(t,z)+(\hat{Y}_t^1-Y_t^1)(-\frac{\partial\hat{H}}{\partial y_1}(t,z))\\
&+(\hat{Y}_t^2-Y_t^2)(-\frac{\partial\hat{H}}{\partial y_2}(t,z))+(\hat{X}_t-X_t)(-\frac{\partial\hat{H}}{\partial x}(t,z))+(\hat{\phi}(t,z)-\phi(t,z))(-\frac{\partial\hat{H}}{\partial\phi}(t,z)-L^*\hat{p}_4(t,z)\\
&+\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right))+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))\hat{q}_1(t,z)+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))\hat{q}_1'(t,z)+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))\hat{q}_2(t,z)\\
&+(\hat{\sigma}_{22}(t)-\sigma_{22}(t))\hat{q}_2'(t,z)+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))\hat{q}_3(t,z)+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))\hat{q}_3'(t,z)\\
&+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))\hat{q}_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))\hat{q}_4'(t,z)\Big\}dtdz\Big]
\end{aligned}\tag{28}$$

From (27) and (28) we have :

$$
\begin{aligned}
J(\hat{u})-J(u)\geq & I_1+I_2\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{b}_1(t)-b_1(t)\big)\hat{p}_1(t,z)-\big(\hat{b}_2(t)-b_2(t)\big)\hat{p}_2(t,z)\right.\\
&-\big(\hat{b}_3(t)-b_3(t)\big)\hat{p}_3(t,z)-\big(\hat{b}_4(t,z)-b_4(t,z)\big)\hat{p}_4(t,z)-\big(\hat{\sigma}_{11}(t)-\sigma_{11}(t)\big)\hat{q}_1(t,z)\\
&-\big(\hat{\sigma}_{21}(t)-\sigma_{21}(t)\big)\hat{q}_2(t,z)-\big(\hat{\sigma}_{31}(t)-\sigma_{31}(t)\big)\hat{q}_3(t,z)-\big(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z)\big)\hat{q}_4(t,z)\\
&-\big(\hat{\sigma}_{12}(t)-\sigma_{12}(t)\big)\hat{q'}_1(t,z)-\big(\hat{\sigma}_{22}(t)-\sigma_{22}(t)\big)\hat{q'}_2(t,z)-\big(\hat{\sigma}_{32}(t)-\sigma_{32}(t)\big)\hat{q'}_3(t,z)\\
&-\big(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z)\big)\hat{q'}_4(t,z)+\big(\hat{b}_1(t)-b_1(t)\big)\hat{p}_1(t,z)+\big(\hat{b}_2(t)-b_2(t)\big)\hat{p}_2(t,z)\\
&+\big(\hat{b}_3(t)-b_3(t)\big)\hat{p}_3(t,z)+\big(\hat{b}_4(t)-b_4(t)\big)\hat{p}_4(t,z)+L\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)\\
&-\big(\hat{Y}^1_t-Y^1_t\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}^2_t-Y^2_t\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)-\big(\hat{X}_t-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)\\
&-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\\
&+\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))q_1(t,z)+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))q_1'(t,z)\\
&+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))q_2(t,z)+((\hat{\sigma}_{22}(t)-\sigma_{22}(t))q_2'(t,z)+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))q_3(t,z)\\
&\left.+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))q_3'(t,z)+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))q_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))q_4'(t,z)\Big\}dtdz\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{Y}_1(t)-Y_1(t)\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}_2(t)-Y_2(t)\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)\right.\\
&-\big(\hat{X}(t)-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)+\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right.\\
&\left.+L\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\Big\}dtdz\right]\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{Y}_1(t)-Y_1(t)\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}_2(t)-Y^2_t\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)\right.\\
&-\big(\hat{X}(t)-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)+\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)\\
&\left.+L^*\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\Big\}dtdz\right]\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\frac{\partial\hat{H}}{\partial y_1}(t,z)\big(\hat{Y}^1_t-Y^1_t\big)-\frac{\partial\hat{H}}{\partial y_2}(t,z)\big(\hat{Y}^2_t-Y^2_t\big)-\frac{\partial\hat{H}}{\partial x}(t,z)\big(\hat{X}_t-X_t\big)\right.\\
&\left.-\frac{\partial\hat{H}}{\partial\phi}(t,z)\big(\hat{\phi}(t,z)-\phi(t,z)\big)-\frac{\partial\hat{H}}{\partial\phi'}(t,z)\big(\hat{\phi}'(t,z)-\phi'(t,z)\big)\Big\}dtdz\right]
\end{aligned}
\tag{29}
$$

!

. Recall that: $H(t,z)=H(t,z,Y^1_t,Y^2_t,X_t,u_t,\phi(t,z),\phi'(t,z),\hat{p},\hat{q},\hat{q'})$,
$\hat{H}(t,z)=H(t,z,\hat{Y}^1_t,\hat{Y}^2_t,\hat{X}_t,\hat{u}_t,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{p},\hat{q},\hat{q'})$,
$H(t,z,X,Y^1,Y^2,\phi,\phi',\hat{u},\hat{p},\hat{q})=\sup_{u\in U_{ad}}H(t,z,X,Y^1,Y^2,\phi,\phi',u,\hat{p},\hat{q})\quad\forall X,Y^1,Y^2,\phi,\phi'$
$\hat{H}(t,z)=\sup_{u\in U_{ad}}H(t,z,\hat{X},\hat{Y}^1,\hat{Y}^2,\hat{\phi},\hat{\phi}',u,\hat{p},\hat{q})$; $\hat{H}(t,z)=H(t,z,\hat{X},\hat{Y}^1,\hat{Y}^2,\hat{\phi},\hat{\phi}',\hat{u},\hat{p},\hat{q})$,

According to (29) and the concavity of *h* we have :

$$
\begin{aligned}
J(\hat{u})-J(u) \geq & \mathbb{E}\Bigg[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z,X,Y^1,Y^2,\phi,\phi^{'},\hat{u},\hat{p},\hat{q})-\frac{\partial\hat{H}}{\partial y_1}(t,z)\big(\hat{Y}^1_t-Y^1_t\big)\\
&-\frac{\partial\hat{H}}{\partial y_2}(t,z)\big(\hat{Y}^2_t-Y^2_t\big)-\frac{\partial\hat{H}}{\partial x}(t,z)\big(\hat{X}_t-X_t\big)-\frac{\partial\hat{H}}{\partial\phi}(t,z)\big(\hat{\phi}(t,z)-\phi(t,z)\big)\\
&-\frac{\partial\hat{H}}{\partial\phi'}(t,z)\big(\hat{\phi}'(t,z)-\phi(t,z)\big)\Big\}dtdz\Bigg]\\
\geq & 0
\end{aligned} \tag{30}
$$

□

5.Optimal investment

We obtain from section 3:

$$
\begin{cases}
dS_t=\frac{\sigma_1^2 S_t}{2}dt+\sigma_1 S_t d\tilde{Y}^1_t, & S_0=s\\
dr_t=\sigma_2\sqrt{r_t}d\tilde{Y}^2_t, & r_0, \qquad \sigma_2^2<2ab\\
dX^{x,u}_t=\left[r_tX^{x,u}_t-u_t(r_t-\frac{1}{2}\sigma_1^2)\right]dt+u_t\sigma_1 d\tilde{Y}^1_t, & X^{x,u}_0=x\\
d\phi(t,z)=\left[\sigma_3^2\dfrac{\partial^2\phi}{\partial^2 z}(t,z)-k(\beta-z)\dfrac{\partial\phi}{\partial z}(t,z)+k\phi(t,z)\right]dt & \\
\qquad +\left[\mu(r_t,z)-\dfrac{1}{2}\sigma_1^2-\rho_1\sigma_3\dfrac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}^1_t+\left[a(b-r_t)-\sigma_3\rho_2\dfrac{\partial\phi}{\partial z}(t,z)\right]d\tilde{Y}^2_t & \phi(0,z)=\xi(z)\\
\qquad =L^*_Z\phi(t,z)dt+M^*_1\phi(t,z)d\tilde{Y}^1_t+M^*_2\phi(t,z)d\tilde{Y}^2_t &
\end{cases} \tag{31}
$$

We find that only the process $(X^{x,u}_t)$ depends on the control u.

$$
\begin{aligned}
V_\Pi(x)&=\sup_{u\in U_{ad}}\mathbb{E}[U\left(X^{x,u}_T+\Pi(S_T,B)\right)]\\
&=\sup_{u\in U_{ad}}\tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}U\left(X^{x,u}_T+\Pi(S_T,B(z)+\bar{b})\right)\phi(T,z)d\mathbb{P}_{\bar{B}}dz\right]\\
&=\tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}U\left(X^{x,\hat{u}}_T+\Pi(S_T,B(z)+\bar{b})\right)\phi(T,z)d\mathbb{P}_{\bar{B}}dz\right]
\end{aligned} \tag{32}
$$

the objective function to be maximized becomes:

$$
J(u)=\tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\left[U\left(X^{x,u}_T+\Pi(S_T,B(z)+\bar{b})\right)\right]\phi(T,z)d\mathbb{P}_{\bar{B}}dz\right]
$$

Let $\phi''(t,z)=\frac{\partial^2\phi}{\partial^2 z}(t,z),\phi'(t,z)=\frac{\partial\phi}{\partial z}(t,z)$. Applying Theorem 4.1 with the Hamiltonian:

$$
\begin{aligned}
H(t,z,s,r,x,\phi,\phi',u,p,q,q')=&\frac{\sigma_1^2 s}{2}p_1+\left[rx-u(r-\frac{1}{2}\sigma_1^2)\right]p_3+\left[-k(\beta-z)\phi'+k\phi\right]p_4+\sigma_1 sq_1+\sigma_1 uq_3\\
&+\left[\mu(r,z)-\frac{1}{2}\sigma_1^2-\rho_1\sigma_3\phi'\right]q_4+\sigma_2\sqrt{r}q_2'+\left[a(b-r_t)-\sigma_3\rho_2\phi'\right]q_4'
\end{aligned} \tag{33}
$$

.The adjoining equations are given by:

$$
\begin{cases}
-dp_1(t,z) &=\left[\frac{\sigma_1^2}{2}p_1(t,z)+\sigma_1 q_1(t,z)\right]dt-q_1(t,z)d\tilde{Y}^1_t-q_1'(t,z)d\tilde{Y}^2_t\\
p_1(T,z) &=\phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial S}d\mathbb{P}_{\bar{B}}
\end{cases} \tag{34}
$$

$$
\begin{cases}
-dp_2(t,z) &=\left[(x-u)p_3(t,z)+\frac{\partial\mu}{\partial r}(r,z)q_4(t,z)+\frac{\sigma_2}{2\sqrt{r}}q_2'(t,z)-aq_4'(t,z)\right]dt-q_2(t,z)d\tilde{Y}^1_t-q_2'(t,z)d\tilde{Y}^2_t\\
p_2(T,z) &=\phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial r}d\mathbb{P}_{\bar{B}}
\end{cases} \tag{35}
$$

$$
\begin{cases}
-dp_{3(t,z)} &=rp_3(t,z)dt-q_3(t,z)d\tilde{Y}^1_t-q_3'(t,z)d\tilde{Y}^2_t\\
p_3(T,z) &=\phi(T,z)\int_{\mathbb{R}}\frac{\partial U}{\partial x}d\mathbb{P}_{\bar{B}}
\end{cases} \tag{36}
$$

$$\begin{cases} -dp_4(t,z) & = \left[\sigma_3^2 \frac{\partial^2}{\partial^2 z} p_4(t,z)\right] dt - q_4(t,z) d\tilde{Y}_t^1 - q_4'(t,z) d\tilde{Y}_t^2 \\ p_4(T,z) & = \int_{\mathbb{R}} U d\mathbb{P}_{\bar{B}} \end{cases} \quad (37)$$

The generators of some of these equations are linear in their arguments. There is existence and uniqueness of the solutions of these BSDE in a suitable space of Banach. Moreover, the last equation which is an BSPDE is classical thus, let μ such that (18), (34)-(37) have a unique strong solution; see for example [3].
Let $\hat{u}$ be an optimal control with the corresponding optimal processes $\hat{X}^{x,\hat{u}}$, $\hat{S}$, $\hat{r}$, $\hat{\phi}$, $\hat{\phi}'$ and the corresponding adjoining equations with solutions: $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4, \hat{q}_1', \hat{q}_2', \hat{q}_3', \hat{q}_4'$.
U and Π are concave. It follows that the first condition of Theorem 4.1 is satisfied. H is a linear function of u. So the first order optimality condition below:

First order optimality condition:

$$\frac{\partial H}{\partial u}(t,z) = 0 \Longleftrightarrow (r_t - \frac{1}{2}\sigma_1^2)\hat{p}_3(t,z) = \sigma_1 \hat{q}_3(t,z) \quad (38)$$

The equation satisfied by p_3 is linear.

$$\begin{cases} -d\hat{p}_{3(t,z)} & = r\hat{p}_3(t,z)dt - \hat{q}_3(t,z)d\tilde{Y}_t^1 - \hat{q}_3'(t,z)d\tilde{Y}_t^2 \\ \hat{p}_3(T,z) & = \phi(T,z)\int_{\mathbb{R}} \frac{\partial U}{\partial x} d\mathbb{P}_{\bar{B}} \end{cases} \quad (39)$$

Let us find a solution of (39) in the form

$$\hat{p}_3(t,z) = \rho(t, r_t, \phi(t,z)) X_t^{x,\hat{u}} + \psi(t, S_t, r_t, \phi_t) \quad (40)$$

where ψ is a C^2 function in each of its variables with the differential form:

$$d\psi(t, S_t, r_t, \phi_t) = \Theta_t dt + \beta_t^1 d\tilde{Y}_2^1 + \beta_t^2 d\tilde{Y}_2^2 \quad (41)$$

$$\begin{aligned} d\rho(t, r_t, \phi(t,z)) = & \Bigg[\frac{\partial \rho}{\partial t} + \left(\sigma_3^2 \frac{\partial^2 \phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right)\frac{\partial \rho}{\partial \phi} + \frac{1}{2}\sigma_2^2 r_t \frac{\partial^2 \rho}{\partial^2 r} \\ & + \frac{1}{2}\left[\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)^2 + \left(a(b - r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)^2\right]\frac{\partial^2 \rho}{\partial^2 \phi} \\ & + \sigma_2\sqrt{r_t}\left(a(b - r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial^2 \rho}{\partial r \partial \phi}\Bigg] dt + \left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi} d\tilde{Y}_t^1 \\ & + \left[\sigma_2\sqrt{r_t}\frac{\partial \rho}{\partial r} + \left(a(b - r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}\right] d\tilde{Y}_t^2 \end{aligned}$$

. (42)

We have:

$$\begin{aligned} d\hat{p}_3(t,z) = & d\rho(t, r_t, \phi(t,z))X_t^{x,\hat{u}} + \rho(t, r_t, \phi(t,z))dX_t^{x,\hat{u}} + \sigma_1\hat{u}_t\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)\frac{\partial \rho}{\partial \phi}dt \\ & + \Theta_t dt + \beta_t^1 d\tilde{Y}_t^1 + \beta_t^2 d\tilde{Y}_t^2 \end{aligned}$$

.

$$
\begin{aligned}
d\hat{p}_3(t,z) = \Bigg[\Bigg[&\frac{\partial\rho}{\partial t} + \left(\sigma_3^2\frac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2\rho}{\partial^2 r} \\
&+ \frac{1}{2}\left(\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right)^2 + \left(a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)^2\right)\frac{\partial^2\rho}{\partial^2\phi} \\
&+ \sigma_2\sqrt{r_t}\left(a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)\frac{\partial^2\rho}{\partial r\partial\phi} + r_t\rho(t,r_t,\phi(t,z))\Bigg]X_t^{x,\hat{u}} - \rho(t,r_t,\phi(t,z))\hat{u}_t(r_t - \frac{1}{2}\sigma_1^2) \\
&+ \sigma_1\hat{u}_t\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right)\frac{\partial\rho}{\partial\phi} + \Theta_t\Bigg]dt \\
&+ \left[\left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial\rho}{\partial\phi}X_t^{x,\hat{u}} + \rho(t,r_t,\phi(t,z))\hat{u}_t\sigma_1 + \beta_t^1\right]d\tilde{Y}_t^1 \\
&+ \left[\left[\sigma_2\sqrt{r_t}\frac{\partial\rho}{\partial r} + \left(a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)\frac{\partial\rho}{\partial\phi}\right]X_t^{x,\hat{u}} + \beta_t^2\right]d\tilde{Y}_t^2.
\end{aligned}
\tag{43}
$$

On the other hand, according to (39) and (40) , we have:

$$
d\hat{p}_3(t,z) = -r_t\left[\rho(t,r_t,\phi(t,z))X_t^{x,\hat{u}} + \psi(t,S_t,r_t,\phi_t)\right]dt + \hat{q}_3 d\tilde{Y}_t^1 + \hat{q}_3' d\tilde{Y}_t^2 \tag{44}
$$

By identification of (43) and (44) we have:

$$
\begin{cases}
\Bigg[\frac{\partial\rho}{\partial t} + \left(\sigma_3^2\frac{\partial^2\phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial\phi}{\partial z}(t,z) + k\phi(t,z)\right)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2\rho}{\partial^2 r} \\
+\frac{1}{2}\left(\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right)^2 + \left(a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)^2\right)\frac{\partial^2\rho}{\partial^2\phi} \\
+ \sigma_2\sqrt{r_t}\left(a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)\frac{\partial^2\rho}{\partial r\partial\phi} + r_t\rho(t,r_t,\phi(t,z))\Bigg]X_t^{x,\hat{u}} - \rho(t,r_t,\phi(t,z))\hat{u}_t(r_t - \frac{1}{2}\sigma_1^2) \\
+\sigma_1\hat{u}_t\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right)\frac{\partial\rho}{\partial\phi} + \Theta_t \\
\quad = -r_t\rho(t,r_t,\phi(t,z))X_t^{x,\hat{u}} - r_t\psi(t,S_t,r_t,\phi(t,z)) & \text{(45a)} \\
\left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial\phi}{\partial z}(t,z)\right]\frac{\partial\rho}{\partial\phi}X_t^{x,\hat{u}} + \rho(t,r_t,\phi(t,z))\hat{u}_t\sigma_1 + \beta_t^1 = \hat{q}_3(t,z) & \text{(45b)} \\
\left[\sigma_2\sqrt{r_t}\frac{\partial\rho}{\partial r} + \left(a(b - r_t) - \sigma_3\rho_2\frac{\partial\phi}{\partial z}(t,z)\right)\frac{\partial\rho}{\partial\phi}\right]X_t^{x,\hat{u}} + \beta_t^2 = \hat{q}_3'(t,z).
\end{cases}
$$

$$
\text{(45a)} \Updownarrow
$$

$$
\hat{u}_t = \frac{\begin{aligned}&\left[\tfrac{\partial\rho}{\partial t} + L_Z^*\phi(t,z)\tfrac{\partial\rho}{\partial\phi} + \tfrac{1}{2}\sigma_2^2 r_t\tfrac{\partial^2\rho}{\partial^2 r} + \tfrac{1}{2}\left((M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right)\tfrac{\partial^2\rho}{\partial^2\phi}\right. \\ &\left. + \sigma_2\sqrt{r_t}M_2^*\phi(t,z)\tfrac{\partial^2\rho}{\partial r\partial\phi} + 2r_t\rho(t,r_t,\phi(t,z))\right]X_t^{x,\hat{u}} + r_t\psi(t,S_t,r_t,\phi_t) + \Theta_t\end{aligned}}{\rho(t,r_t,\phi(t,z))\left(r_t - \frac{1}{2}\sigma_1^2\right) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}} . \tag{46}
$$

We have by (45b) and the first order optimality condition (38):

$$
\hat{u}_t = \frac{\left[\left(r_t - \frac{1}{2}\sigma_1^2\right)\rho(t,r_t,\phi(t,z) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}\right]X_t^{x,\hat{u}} + \left(r_t - \frac{1}{2}\sigma_1^2\right)\psi(t,S_t,r_t,\phi_t) - \sigma_1\beta_t^1}{\sigma_1^2\rho(t,r_t,\phi(t,z))} \tag{47}
$$

The equations satisfied by $\rho(t, r_t, \phi(t,z))$ *and* $\psi(t, S_t, r_t, \phi_t)$ are obtained by identifying (46) and (47) as a first degree polynomial in $X_t^{x,\hat{u}}$.

Thus, we have

(48a)

$$\begin{cases} \dfrac{\frac{\partial\rho}{\partial t} + L_Z^*\phi(t,z)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2 r_t \frac{\partial^2\rho}{\partial^2 r} + \frac{1}{2}\left((M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right)\frac{\partial^2\rho}{\partial^2\phi} + \sigma_2\sqrt{r_t}M_2^*\phi(t,z)\frac{\partial^2\rho}{\partial r\partial\phi} + 2r_t\rho(t,r_t,\phi(t,z))}{\rho(t,r_t,\phi(t,z))\left(r_t - \frac{1}{2}\sigma_1^2\right) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}} \\ = \dfrac{(r_t - \frac{1}{2}\sigma_1^2)\rho(t,r_t,\phi(t,z)) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1^2\rho(t,r_t,\phi(t,z))} \\ \dfrac{\Theta_t + r_t\psi(t,S_t,r_t,\phi_t)}{\rho(t,r_t,\phi(t,z))\left(r_t - \frac{1}{2}\sigma_1^2\right) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}} = \dfrac{\left(r_t - \frac{1}{2}\sigma_1^2\right)\psi(t,S_t,r_t,\phi_t) - \sigma_1\beta_t^1}{\sigma_1^2\rho(t,r_t,\phi(t,z))} \end{cases}$$

(48b)

$$(48a)$$
$$\Updownarrow$$

$$\frac{\partial\rho}{\partial t} + L_Z^*\phi(t,z)\frac{\partial\rho}{\partial\phi} + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2\rho}{\partial^2 r} + \frac{1}{2}\left((M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right)\frac{\partial^2\rho}{\partial^2\phi} + \sigma_2\sqrt{r_t}M_2^*\phi(t,z)\frac{\partial^2\rho}{\partial r\partial\phi}$$
$$+ 2r_t\rho(t,r_t,\phi(t,z)) = \left(\frac{(r_t - \frac{1}{2}\sigma_1^2)}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r_t,\phi(t,z))}\right)\left[(r_t - \frac{1}{2}\sigma_1^2)\rho(t,r_t,\phi(t,z)) - \sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}\right] \tag{49}$$

(49) is a parabolic PDE. With quadratic utility function $U(x) = -(x - \alpha_0)^2$ and terminal condition:

$$\rho(T, r_T, \phi(T,z)) = -2\,\phi(T,z). \qquad (50)$$

If for example we take $\frac{\partial\rho}{\partial\phi} = 0$, by Feynman-Kac representation formula, we have ρ solution of backward stochastic differential equation:

$$dY_t = \left[\left(\frac{(r_t - \frac{1}{2}\sigma_1^2)^2}{\sigma_1^2} - 2r_t\right)Y_t\right]dt + Z_t^2 d\tilde{Y}_t^2, \quad Y_T = -2\phi(T,z) \tag{51}$$

$$\rho(t,r) = \exp\left(\int_0^t\left(\frac{(r_s - \frac{1}{2}\sigma_1^2)^2}{\sigma_1^2} - 2r_s\right)ds\right)\mathbb{E}\left[-2\phi(T,z)\exp\left(-\int_0^T\left[\left(\frac{r_s - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 - 2r_s\right]ds\right)|r_t = r\right] \tag{52}$$

Equation of ψ

$$(48b)$$
$$\Updownarrow$$

$$r_t\psi(t,S_t,r_t,\phi(t,z)) + \Theta_t = \left[\frac{(r_t - \frac{1}{2}\sigma_1^2)}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r_t,\phi(t,z))}\right]\left[(r_t - \frac{1}{2}\sigma_1^2)\psi(t,S_t,r_t,\phi(t,z)) - \sigma_1\beta_t^1\right]$$
$$\Theta_t + \left[\frac{(r_t - \frac{1}{2}\sigma_1^2)}{\sigma_1} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r_t,\phi(t,z))}\right]\beta_t^1 = \left[\frac{(r_t - \frac{1}{2}\sigma_1^2)^2}{\sigma_1^2} - \frac{(r_t - \frac{1}{2}\sigma_1^2)M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r_t,\phi(t,z))} - r_t\right]\psi(t,S_t,r_t,\phi(t,z)) \tag{53}$$

Let's calculate Θ_t and β^1.

$$
\begin{aligned}
&d\psi(t,S_t,r_t,\phi(t,z)) = \\
&\frac{\partial \psi}{\partial t}(t,S_t,r_t,\phi(t,z))dt + \left[\frac{\sigma_1^2 S_t}{2}dt + \sigma_1 S_t d\tilde{Y}_t^1\right]\frac{\partial \psi}{\partial S}(t,S_t,r_t,\phi(t,z)) \\
&+ \sigma_2\sqrt{r_t}\frac{\partial \psi}{\partial r}(t,S_t,r_t,\phi(t,z))d\tilde{Y}_t^2 + \left[\left[\sigma_3^2\frac{\partial^2 \phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right]dt\right. \\
&\left. + \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right]d\tilde{Y}_t^1 + \left[a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right]d\tilde{Y}_t^2\right]\frac{\partial \psi}{\partial \phi}(t,S_t,r_t,\phi(t,z)) \\
&+ \frac{1}{2}\sigma_1^2 S_t^2\frac{\partial^2 \psi}{\partial^2 S}(t,S_t,r_t,\phi(t,z))dt + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2 \psi}{\partial^2 r}(t,S_t,r_t,\phi(t,z))dt \\
&+ \frac{1}{2}\left[\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)^2 + \left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)^2\right]\frac{\partial^2 \psi}{\partial^2 \phi}(t,S_t,r_t,\phi(t,z))dt \\
&+ \sigma_1 S_t\left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial^2 \psi}{\partial S\partial \phi}(t,S_t,r_t,\phi(t,z))dt \\
&+ \sigma_2\sqrt{r_t}\left[a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial^2 \psi}{\partial r\partial \phi}(t,r_t,r_t,\phi(t,z))dt
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\Theta_t &= \frac{\partial \psi}{\partial t}(t,S_t,r_t,\phi(t,z)) + \frac{\sigma_1^2 S_t}{2}\frac{\partial \psi}{\partial S}(t,S_t,r_t,\phi(t,z)) \\
&\quad + \left[\sigma_3^2\frac{\partial^2 \phi}{\partial^2 z}(t,z) - k(\beta - z)\frac{\partial \phi}{\partial z}(t,z) + k\phi(t,z)\right]\frac{\partial \psi}{\partial \phi}(t,S_t,r_t,\phi(t,z)) \\
&\quad + \frac{1}{2}\sigma_1^2 S_t^2\frac{\partial^2 \psi}{\partial^2 S}(t,S_t,r_t,\phi(t,z)) + \frac{1}{2}\sigma_2^2 r_t\frac{\partial^2 \psi}{\partial^2 r}(t,S_t,r_t,\phi(t,z)) \\
&\quad + \frac{1}{2}\left[\left(\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right)^2 + \left(a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right)^2\right]\frac{\partial^2 \psi}{\partial^2 \phi}(t,S_t,r_t,\phi(t,z)) \\
&\quad + \sigma_1 S_t\left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial^2 \psi}{\partial S\partial \phi}(t,S_t,r_t,\phi(t,z)) \\
&\quad + \sigma_2\sqrt{r_t}\left[a(b-r_t) - \sigma_3\rho_2\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial^2 \psi}{\partial r\partial \phi}(t,S_t,r_t,\phi(t,z))
\end{aligned}
$$

$$
\beta_t^1 = \sigma_1 S_t\frac{\partial \psi}{\partial S}(t,S_t,r_t,\phi(t,z)) + \left[\mu(r_t,z) - \frac{1}{2}\sigma_1^2 - \rho_1\sigma_3\frac{\partial \phi}{\partial z}(t,z)\right]\frac{\partial \psi}{\partial \phi}(t,S_t,r_t,\phi(t,z))
$$

Let $\phi''(t,z) = \frac{\partial^2\phi}{\partial^2 z}(t,z), \phi'(t,z) = \frac{\partial\phi}{\partial z}(t,z)$.

After calculations, we obtain with utility function $U(x) = -(x - \alpha_0)^2$, ψ solution of the PDE:

$$
\begin{aligned}
&\frac{\partial \psi}{\partial t}(t,S,r,\phi) + \frac{\sigma_1^2 S^2}{2}\frac{\partial^2 \psi}{\partial^2 S}(t,S,r,\phi) + \frac{\sigma_2^2 r}{2}\frac{\partial^2 \psi}{\partial^2 r}(t,S,r,\phi) + \frac{1}{2}\left[(M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right]\frac{\partial^2 \psi}{\partial^2 \phi}(t,S,r,\phi) \\
&+ \sigma_1 S M_1^*\phi(t,z)\frac{\partial^2 \psi}{\partial S\partial \phi}(t,S,r,\phi) + \sigma_2\sqrt{r}M_2^*\phi(t,z)\frac{\partial^2 \psi}{\partial r\partial \phi}(t,S,r,\phi) + S\left[r - \frac{\sigma_1 M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}\right]\frac{\partial \psi}{\partial S}(t,S,r,\phi) \\
&+ \left[L_Z^*\phi(t,z) + M_1^*\phi(t,z)\left[\frac{(r - \frac{1}{2}\sigma_1^2)}{\sigma_1} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}\right]\right]\frac{\partial \psi}{\partial \phi}(t,S,r,\phi) \\
&= \left[\frac{\left(r - \frac{1}{2}\sigma_1^2\right)^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)(r - \frac{1}{2}\sigma_1^2)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r,\phi)} - r\right]\psi(t,S,r,\phi)
\end{aligned} \tag{55}
$$

with terminal condition

$$
\psi(T,S,r,\phi) = -2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T, B(z) + \bar{b}) - \alpha_0\right)d\mathbb{P}_{\bar{b}} \tag{56}
$$

Let $M_\rho^*\phi(t,z) = \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}$

. Considering the elliptic operator associated with the state processes given by:

$$\mathcal{L}_\phi^* = \frac{1}{2}\left[(M_1^*\phi)^2 + (M_2^*\phi)^2\right]\partial_{\phi\phi}^2 + \left[L_Z^*\phi + M_1^*\phi\left(\frac{(r - \frac{1}{2}\sigma_{1)}^2}{\sigma_1} - M_\rho^*\phi\right)\right]\partial_\phi \tag{57}$$

(55) becomes:

$$\begin{aligned}\psi_t + \frac{\sigma_1^2 S^2}{2}\psi_{SS} + \frac{\sigma_2^2 r}{2}\psi_{rr} + < \mathcal{L}_\phi^*, \psi > + < M_1^*\phi,\ \sigma_1 S D_\phi \psi_S > + < M_2^*\phi,\ \sigma_2\sqrt{r}D_\phi\psi_r > \\ + < S(r - \sigma_1 M_\rho^*\phi),\ \psi_S > + < r - \left(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi, \psi > = 0\end{aligned} \tag{58}$$

With D_ϕ the partial derivative with respect to ϕ.

By Feynman-Kac representation formula, under regularity assumptions we have ψ solution of BSDEs:

$$\begin{aligned}Y_t^{s,S,r,\phi} =& -2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T, B(z) + \bar{b}) - \alpha_0\right)d\mathbb{P}_{\bar{b}} \\ &+ \int_t^T \left[\left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi)\right)Y_u^{s,S,r,\phi} + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)Z_u^1\right]du \\ &- \int_t^T Z_u^1 d\tilde{Y}_u^1 - \int_t^T Z_u^2 d\tilde{Y}_u^2, \quad 0 \le t \le T.\end{aligned} \tag{59}$$

Where $S = S_t, r = r_t,\ \phi = \phi(t,z),\ \psi(t, S_t, r_t, \phi_t) = Y_t^{t,S,r,\phi}$

$$\begin{aligned}\psi(t,S,r,\phi) =& \exp\left\{\int_0^t \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi)\right)du - \frac{1}{2}\int_0^t\left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 du \right. \\ &\left. + \int_0^t \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)d\tilde{Y}_t^1\right\} \times \tilde{\mathbb{E}}\left[\left(-2\int_{\mathbb{R}}\phi(T,z)\left(\Pi(S_T, B(z) + \bar{b}) - \alpha_0\right)d\mathbb{P}_{\bar{b}}\right)\right. \\ &\exp\left\{\int_0^T \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)M_\rho^*\phi)\right)du - \frac{1}{2}\int_0^T\left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 du\right. \\ &\left.\left. + \int_0^T\left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)d\tilde{Y}_t^1\right\} | S_t = S, r_t = r, \phi(t,z) = \phi\right]\end{aligned} \tag{60}$$

Optimal control in the presence of the option is given by:

$$
\begin{aligned}
\hat{u}_t =& \frac{\left(r_t - \frac{1}{2}\sigma_1^2\right)}{\sigma_1^2 \rho(t, r_t, \phi(t,z))} \tilde{\mathbb{E}}\Bigg[\Bigg(-2\int_{\mathbb{R}} \phi(T,z)\left(\Pi(S_T, B(z)+\bar{b}) - \alpha_0\right) d\mathbb{P}_{\bar{b}}\Bigg) \\
& \exp\Bigg\{\int_t^T \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t,r,\phi(t,z))}{\rho(t,r,\phi(t,z))}\right) du \\
& -\frac{1}{2}\int_t^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t,r,\phi(t,z))}{\rho(t,r,\phi(t,z))}\right)^2 du \\
& + \int_t^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t,r,\phi(t,z))}{\rho(t,r,\phi(t,z))}\right) d\tilde{Y}_t^1\Bigg\} | S_t = S, r_t = r, \phi(t,z) = \phi\Bigg] \\
& - \frac{1}{\sigma_1\rho(t, r_t, \phi(t,z))}\left[\sigma_1 S_t \frac{\partial\psi}{\partial S}(t, S_t, r_t, \phi(t,z)) + M_1^*\phi(t,z)\frac{\partial\psi}{\partial\phi}(t, S_t, r_t, \phi(t,z))\right] \\
& + \left[\frac{\left(r_t - \frac{1}{2}\sigma_1^2\right)}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t, r_t, \phi(t,z))}{\sigma_1\rho(t, r_t, \phi(t,z))}\right] X_t^{x,\hat{u}} \qquad \forall t \in [0,T]
\end{aligned}
\tag{61}
$$

In the absence of the option, $\Pi(S_T, B(z_T) + \bar{B}) = 0$ and the function ψ does not depend on S_t therefore, the optimal control is given by:

$$
\begin{aligned}
\hat{u}_t^0 =& \frac{\left(r_t - \frac{1}{2}\sigma_1^2\right)}{\sigma_1^2 \rho(t, r_t, \phi(t,z))} \tilde{\mathbb{E}}\Bigg[\Bigg(-2\int_{\mathbb{R}} \phi(T,z)\left(\Pi(S_T, B(z)+\bar{b}) - \alpha_0\right) d\mathbb{P}_{\bar{b}}\Bigg) \\
& \exp\Bigg\{\int_t^T \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t,r,\phi(t,z))}{\rho(t,r,\phi(t,z))}\right) du \\
& -\frac{1}{2}\int_t^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t,r,\phi(t,z))}{\rho(t,r,\phi(t,z))}\right)^2 du \\
& + \int_t^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t,r,\phi(t,z))}{\rho(t,r,\phi(t,z))}\right) d\tilde{Y}_t^1\Bigg\} | S_t = S, r_t = r, \phi(t,z) = \phi\Bigg] \\
& - \frac{1}{\sigma_1\rho(t, r_t, \phi(t,z))}\left[\sigma_1 S_t \frac{\partial\psi_0}{\partial S}(t, S_t, r_t, \phi(t,z)) + M_1^*\phi(t,z)\frac{\partial\psi_0}{\partial\phi}(t, S_t, r_t, \phi(t,z))\right] \\
& + \left[\frac{\left(r_t - \frac{1}{2}\sigma_1^2\right)}{\sigma_1^2} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}(t, r_t, \phi(t,z))}{\sigma_1\rho(t, r_t, \phi(t,z))}\right] X_t^{x,\hat{u}} \qquad \forall t \in [0,T]
\end{aligned}
\tag{62}
$$

this time with ψ_0 solution of the PDE:

$$
\begin{aligned}
& \frac{\partial\psi_0}{\partial t}(t,S,r,\phi) + \frac{\sigma_2^2 r}{2}\frac{\partial^2\psi_0}{\partial^2 r}(t,S,r,\phi) + \frac{1}{2}\left[(M_1^*\phi(t,z))^2 + (M_2^*\phi(t,z))^2\right]\frac{\partial^2\psi_0}{\partial^2\phi}(t,S,r,\phi) \\
& + \sigma_2\sqrt{r}M_2^*\phi(t,z)\frac{\partial^2\psi}{\partial r\partial\phi}(t,S,r,\phi) + \left[L_Z^*\phi(t,z) + M_1^*\phi(t,z)\left[\frac{\left(r - \frac{1}{2}\sigma_1^2\right)}{\sigma_1} - \frac{M_1^*\phi(t,z)\frac{\partial\rho}{\partial\phi}}{\rho(t,r,\phi(t,z))}\right]\right]\frac{\partial\psi_0}{\partial\phi}(t,S,r,\phi) \\
& = \left[\frac{\left(r - \frac{1}{2}\sigma_1^2\right)^2}{\sigma_1^2} - \frac{M_1^*\phi(t,z)(r - \frac{1}{2}\sigma_1^2)\frac{\partial\rho}{\partial\phi}}{\sigma_1\rho(t,r,\phi)} - r\right]\psi_0(t,S,r,\phi)
\end{aligned}
\tag{63}
$$

with terminal condition

$$
\psi^0(T, S, r, \phi) = 2\alpha_0\phi(\mathrm{T,z})\ ) \tag{64}
$$

Under regularity assumptions we have:

$$\psi_0(t,r,\phi) = \exp\left\{\int_0^t \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^*\phi)\right) du - \frac{1}{2}\int_0^t \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 du \right.$$
$$\left. + \int_0^t \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right) d\tilde{Y}_t^1 \right\} \times \tilde{\mathbb{E}}\left[2\alpha_0\phi(T,z)\right.$$
$$\exp\left\{\int_0^T \left(r_u - \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^*\phi)\right) du - \frac{1}{2}\int_0^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right)^2 du \right.$$
$$\left.\left. + \int_0^T \left(\frac{r_u - \frac{1}{2}\sigma_1^2}{\sigma_1^2} - M_\rho^*\phi\right) d\tilde{Y}_t^1 \right\} | S_t = S, r_t = r, \phi(t,z) = \phi \right] \tag{65}$$

In other words

$$\psi_{0_t} + \frac{\sigma_2^2 r}{2}\psi_{0_{rr}} + < \mathcal{L}_\phi^*, \psi_0 > + < M_2^*\phi,\ \sigma_2\sqrt{r}D_\phi\psi_{0_r} > + < r - \left(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1}\right)^2 + \left(\frac{r - \frac{1}{2}\sigma_1^2}{\sigma_1}\right) M_\rho^*\phi, \psi_0 > = 0 \tag{66}$$

Theorem 5.1. *(Optimal control)*
The optimal control $\hat{\mathrm{u}}$ *(resp* $\hat{u}^0$*) to the terminal utility maximization problem under partial information* (6), (7)*(resp* (8)*) is given by* (61) *and* (55) *(resp by* (62) *and* (63)*).*

6. Conclusions

Information asymmetry and market efficiency today at the basis of modern market finance theory, studied respectively by the economists Akerlof, G Nobel Prize in Economics 2001 and Farma, E Nobel Prize in Economics 2013 raise the question of information and price formation. Unfortunately, the majority of hedging and pricing methods do not take into account the notion of information. Our contribution in this article is multiple. On the theoretical level, we contribute significantly to the use of unobservable variables and to stochastic optimization. On a practical level, in mathematical finance, we highlight the notion of information and the use of stochastic interest rates in the methods of hedging and pricing of contingent assets, mainly European options on incomplete markets.

In this work, we have used the CIR short-term interest rate model and a stochastic maximum principle established with the backward stochastic differential equations for determine the optimal investment strategies in the risky asset constituting the portfolio of coverage of a European option. This is done in the presence of the option and in its absence. We assumed that the payoff of the option does not depend only on the price of the risky asset on the maturity date. But it also depends on uncertain events related to the market environment. The passage from partial to complete information was done using Girsanov's theorem and Zakai's equation. We have assumed that the management portfolio used for hedging of this option consists of a risky financial asset and an asset that we have qualified as semi-risky whose price is a deterministic function of the short-term interest rate that we have assumed to be stochastic. We used as an interest rate model the CIR model which particularity of not giving negative values. We took as observable variables the price of the risky asset, the short-term interest rate and like the unobservable one, a variable representing all uncertain events related to the market environment with the maximum principle established. we were able to solve the problem of coverage of this contingent asset on risky asset. We have thus given the optimal amount to invest in risky assets at all times. As prospects, we intend to generalize these results to the management of a portfolio made up of several financial assets and do digital simulations.

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